Comment on "Relationships for Motor Temperature Sensitivity"

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REFERENCE 1 presents an analysis of motor temperature sensitivity relations that concludes ($n = [\partial \ln r/\partial \ln p]_T$) (Reference 1 nomenclature is employed, and equations and figures are numbered to correspond with it):

$$\pi_K = \frac{\left(\dfrac{\partial \ln c}{\partial T}\right)_K + \ln p \left(\dfrac{\partial n}{\partial T}\right)_K + \pi_C}{1 - n} \tag{10}$$

is generally applicable whereas

$$\pi_K = \frac{\sigma_p + \pi_C}{1 - n} \tag{5}$$

is limited to situations where n is not a function of pressure. However, Ref. 2 showed Eq. (5) applies to any smooth $r(p, T)$, and Refs. 3 and 4 proved Eqs. (5) and (10) are algebraic identities of equal generality. Since Ref. 1 did not address these previous works and Refs. 1 and 4 reach opposite conclusions, which is correct? The objective of this Technical Comment is to examine this situation in detail from Ref. 1's perspective and resolve this question.

Equating Eqs. (10) and (11) shows the requirement for equality to be

$$\left[\frac{\partial \ln c}{\partial T}\right]_K + \ln p \left[\frac{\partial n}{\partial T}\right]_K = \left[\frac{\partial \ln c}{\partial T}\right]_p + \ln p \left[\frac{\partial n}{\partial T}\right]_p \tag{17}$$

Reference 1 claims this requires equality of the derivatives with common dependent variables, viz.

$$\begin{aligned}\left[\frac{\partial \ln c}{\partial T}\right]_K &= \left[\frac{\partial \ln c}{\partial T}\right]_p = \frac{\ln c_2 - \ln c_1}{T_2 - T_1}\\ \left[\frac{\partial n}{\partial T}\right]_K &= \left[\frac{\partial n}{\partial T}\right]_p = \frac{n_2 - n_1}{T_2 - T_1}\end{aligned} \tag{12}$$

However, since $n = n(p, T)$ and $c = c(p, T) > 0$

$$\begin{aligned}\left[\frac{\partial \ln c}{\partial T}\right]_K &= \left[\frac{\partial \ln c}{\partial T}\right]_p + \pi_K \left[\frac{\partial \ln c}{\partial \ln p}\right]_T\\ \left[\frac{\partial n}{\partial T}\right]_K &= \left[\frac{\partial n}{\partial T}\right]_p + \pi_K \left[\frac{\partial n}{\partial \ln p}\right]_T\end{aligned} \tag{18}$$

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Table 1 Variable exponent example ($r = cp^n$)

Rate	$\ln r = \ln T + (0.3 + 0.01 \ln p)\ln p$
n, σ_p	$n = 0.3 + 0.02 \ln p,\ \sigma_p = 1/T$
c	$\ln c = \ln T - 0.01(\ln p)^2$
Derivatives	$\left[\frac{\partial n}{\partial T}\right]_K = 0.02\pi_K,\ \left[\frac{\partial n}{\partial T}\right]_p = 0$
	$\left[\frac{\partial \ln c}{\partial T}\right]_K = \frac{1}{T} - 0.02\pi_K \ln p,\ \left[\frac{\partial \ln c}{\partial T}\right]_p = \frac{1}{T}$
Eq. (17)	$\left(\frac{1}{T} - 0.02\pi_K \ln p\right) + 0.02\pi_K \ln p = \frac{1}{T} + 0$
Eq. (5)	$\pi_K = \frac{1/T + \pi_C}{1 - 0.3 - 0.02 \ln p}$
Eq. (10)	$\pi_K = \frac{1/T - 0.02\pi_K \ln p + \ln p(0.02\pi_K) + \pi_C}{1 - 0.3 - 0.02 \ln p}$

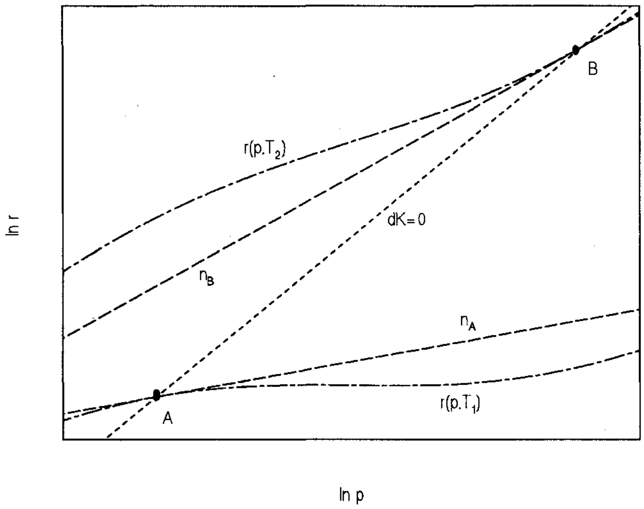

Fig. 3 Sketch illustrating $n(p, T)$ and $n(T)$ situations.

Substituting Eq. (18) into Eq. (17) reduces it to

$$\left[\frac{\partial \ln c}{\partial \ln p}\right]_T + \ln p \left[\frac{\partial n}{\partial \ln p}\right]_T = 0 \tag{19}$$

Restating Eq. (6) as $\ln r = \ln c + n \ln p$ and differentiating with respect to $\ln p$ gives

$$\left[\frac{\partial \ln r}{\partial \ln p}\right]_T = \left[\frac{\partial \ln c}{\partial \ln p}\right]_T + n + \ln p \left[\frac{\partial n}{\partial \ln p}\right]_T = n \tag{20}$$

Comparison of Eqs. (19) and (20) shows that Eq. (17) is unrestricted and that Eq. (12) is not necessary. Moreover, Eq. (12) will generally be incorrect in variable exponent situations.

Table 1 presents a simple variable exponent example. The results show the following in turn—Eq. (17) is correct, Eq. (12) is incorrect for both differential and finite difference forms [Equation (12) equates mean and local sensitivities thereby requiring constant local sensitivities], and Eqs. (5) and (10) are identical. Since $n = n(p)$, Ref. 1's conclusion is clearly incorrect.

Reference 1's formal derivation of Eqs. (15) and (16) equates differential and finite difference π_K. Since this requires $d\pi_K = 0$ along the $dK = 0$ path, Eqs. (15) and (16) are restricted by their derivation to essentially $dn = d\sigma_p = 0$ situations.

However, quasisteady mass conservation at states $A(T_1, K)$ and $B(T_2, K)$ requires

$$\frac{r_B c_B^*}{r_A c_A^*} = \frac{c_B p_B^{n_B} c_B^*}{c_A p_A^{n_A} c_A^*} = \frac{p_B}{p_A} \tag{21}$$

and algebraic rearrangement gives the unrestricted equation for mean π_K (effectively achieved first by Ref. 5)

$$\bar{\pi}_K = \frac{1}{1 - n_B}\left[\frac{\ln(c_B/c_A)}{T_2 - T_1} + \ln p_A \frac{n_B - n_A}{T_2 - T_1} + \pi_C\right] \tag{22}$$

Comparison of Eqs. (15), (16), and (22)[6] shows they have identical form when the initial and final state subscripts correspond. Therefore, the restricted and unrestricted equations are algebraic identities.

Figure 3 illustrates this situation graphically. The figure shows $r(p, T_1)$, $r(p, T_2)$, a $dK = 0$ line, and tangents to the burning rate relations at states A and B (n_A, n_B). Identical $\bar{\pi}_K$ are obtained from either the $n(p, T)$ or $n(T)$ result represented by the tangents, because $\bar{\pi}_K$ is solely a function of the end states. Figure 2 illustrates the identical situation for Eqs. (15) and (16).[6]

In summary, Ref. 1 effectively constrained their mathematics to fit their expectations of Eq. (5) and found results that matched the expectations, but not reality.

References

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